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A POWER SERIES SOLUTION FOR COM-
PRESSIBLE FLOW PAST A
CONICAL SHOCK WAVE

by

Jay A. Leavitt

July 13, 1963

Institute of Mathematical Sciences

NEW YORK UNIVERSITY
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ABSTRACT

The partial differential equations for supersonic conical flow are nonlinear and hyperbolic. By inverting the problem and assuming the shape of the shock, there results a Cauchy problem with initial data given by the Rankine-Hugoniot conditions. This Cauchy problem is then solved numerically by means of a program written for the IBM 7090 Computer. The numerical method was developed by R. D. Richtmyer and G. E. Lewis. Customarily, finite difference schemes are tried to solve these types of problems. However, Richtmyer and Lewis utilized a power series technique suggested by the Cauchy-Kowalewski theory. The numerical procedures involve algorithms which direct the machine to compute the power series, techniques of significance arithmetic, and methods of analytic continuation. The feature of the methods involved is the high degree of accuracy in the computation.

TABLE OF CONTENTS

	Page
Abstract	2
Section	
1. Introduction	4
2. Methods of Continuation	5
3. The Problem	9
4. The Algorithms	16
5. The Body	20
6. The Consistency Checks	21
7. Description of the Calculation	23
8. Results	26
Bibliography	30

Figures and Tables

Figure 1	10
Figure 2	14
Figure 3	22
Figure 4	22
Figure 5	27
Table 1	31
Table 2A	31
Table 2B	33
Table 2C	33
Table 2D	35
Explanation of Tables 2	35

A POWER SERIES SOLUTION FOR COMPRESSIBLE FLOW PAST A CONICAL SHOCK WAVE

by

Jay A. Leavitt

1. Introduction

The usefulness of power series for solving differential equations has been recognized for more than a century. The Cauchy-Kowalewski Theorem proves the existence of a unique analytic solution for noncharacteristic initial value problems and at the same time provides a method for finding a power series solution. However, until recently, this method has been useless as a computational technique except for the simplest of problems. In order to apply the Cauchy-Kowalewski Theorem to a supersonic flow involving a shock, it is first necessary to describe the flow by the "inverse method". In the "inverse method" one assumes the shape of the shock and the conditions ahead of it, and tries to find the flow behind the shock. By appropriate choice of coordinates the shock acts as the initial curve, $\xi = 0$; the Rankine-Hugoniot conditions provide the initial conditions for the system of equations describing the flow. The Cauchy-Kowalewski method is a recursive technique for finding the higher order derivative of the unknowns in terms of the lower order ones. Several different attempts at solving fluid dynamics problems by finding truncated power series failed because of singularities of the solutions. These methods failed to succeed whenever the singularities were closer to the point of expansion than the body. These difficulties were discussed by Van Dyke [1]. In his paper, he states that for the case of axisymmetric flow past a blunt body, there exists a limiting line, a line which is an envelope of

characteristics. The first extensive success using power series methods was initiated by R. D. Richtmyer on the Univac and was later completed by G. Lewis on the I. B. M. 704.[2, 3]. In addition to providing the recursive relations by which the machine could calculate the series coefficients by subroutines Richtmyer's main idea was to carry an accuracy index with each coefficient. Those terms where significance was lost were discarded so that they would not interfere with the calculation. This was done by inspecting the accuracy index as each term was calculated. G. Lewis was able to complete this problem by devising two methods of continuation for truncated power series. The difficulty lay in the fact that a truncated power series is only a polynomial, and to expand it around another point is just a rearrangement of terms, which provides nothing new. The proofs of the convergence of his methods of analytic continuation are found in his thesis [3].

After the bow wave problem was solved, G. Whitham suggested that the method be applied to the conical flow problem. It is to this end that this paper is directed. Solutions to a linearized version of this problem were presented by A. Ferri [4]. Another technique was used by B. Briggs [5, 6], who described the flow in terms of two stream functions. He used a difference method to integrate the resulting equations.

The author wishes to thank R. D. Richtmyer whose encouragement and guidance made this paper possible.

2. Methods of Continuation

In this problem both of Lewis' methods were tried. The second was used in the final computation. The first method was discarded because it was found that in repeated continuations, the coefficients lost most of their significance. In a test where a series was expanded about a point x_0 , and then continued as an expansion about x_1 , it was found

that when this second series was re-expanded around x_0 , about half of the terms were lost due to the loss of significance. The second method proved quite satisfactory, especially since it required no new coding. The program that computed the power series from the initial data was simply reapplied with new initial data obtained from the series to be continued. The second method has the additional advantage of making more use of the differential equations than the first method.

a. First method of continuation

Let $F(x)$ be a function of $x = (x_1, \dots, x_q)$, analytic in each variable, which is expressible as a power series about the origin. Let D_0 be the domain of convergence of this power series and let x^0 be any fixed point in D_0 . $F(x)$ may, then, be expressed by a power series about x^0 . Let D_1 be the domain of convergence of this new power series and let x^1 be any fixed point in D_1 . The problem is to find an approximation to $F(x^1)$ when only the truncated power series about the origin of $F(x)$ is known.

Introduce the new coordinate system

$$\xi = x - x^0 \quad (\xi_i = x_i - x_i^0)$$

and let

$$\xi^0 = x^1 - x^0.$$

Let $f(\xi) = F(\xi + x^0)$, then $F(x^1) = f(\xi^0)$. Denote by $F_k(x)$ the known truncation of the expansion of $F(x)$ about the origin to order k and let $f_k(\xi) = F_k(\xi + x^0)$. Denote by $f_{s,k}(\xi)$ those terms of $f_k(\xi)$ of order less than or equal to $[sk]$ when s is some constant between zero and one ($[sk]$ is the greatest integer in sk). This is a further truncation of $F_k(x)$ written as an expansion of ξ .

Theorem: Under these assumptions, there exists an s , $0 < s < 1$, such that

$$\lim_{k \rightarrow \infty} f_{s,k}(\xi^0) = F(x^1) \quad .$$

The theorem shows that an analytic continuation can be found since $x^1 \in D_1$ is not necessarily in D_0 .

b. Second method of continuation

Let

$$(A) \quad \frac{\partial u}{\partial x} = G(u) \frac{\partial u}{\partial y} \quad \text{and} \quad u(0, y) = f(y) \quad ,$$

where $G(u)$ is analytic at $u = 0$, $G(y) = \sum_{n=0}^{\infty} a_n u^n$. (In the problem,

G also depends explicitly on x and y , but for simplicity assume

$G = G(u)$. The theorem is true for the most general analytic non-characteristic initial value problem.) $f(y)$ is analytic at $y = 0$,

$$f(y) = \sum_{n=0}^{\infty} b_n y^n \quad \text{and only the coefficients of order less than or equal}$$

to k of this expansion of $f(y)$ are known. Let $u_k(x, y)$ be the polynomial of order k obtained from the series expansion of $u(x, y)$ in powers of x and y .

Let (x_0, y_0) be a point in the domain D_0 of absolute convergence of the power series of $u(x, y)$ expanded about the origin, and let (x_1, y_1) be a point in the domain D_1 of absolute convergence of the power series of $u(x, y)$ expanded about the point (x_0, y_0) . The problem is to find an approximation to $u(x_1, y_1)$ when only

$$f_k(y) = \sum_{n=0}^k b_n y^n \quad ,$$

and therefore only $u_k(x, y)$, is known. As in the first method

$(x_1, y_1) \in D_1$ need not be in D_0 . The procedure for finding $u_k(x, y)$ is found in Section 7.

Introduce the new coordinates

$$\begin{aligned}\xi &= x - x_0, & \zeta &= y - y_0 \\ \xi_0 &= x_1 - x_0, & \zeta_0 &= y_1 - y_0\end{aligned}$$

and let

$$\frac{\partial w}{\partial \xi} = G(w) \frac{\partial w}{\partial \zeta}, \quad w(0, \zeta) = u_k(x_0, \zeta + y_0). \quad (B)$$

Also let $w_{s,k}(\xi, \zeta)$ be that polynomial which is obtained from the series expansion of $w(\xi, \zeta)$ in powers of ξ and ζ where all the terms of order greater than $[sk]$ are discarded, s is some constant between zero and one. G. Lewis found that $s = 1$ was apparently best for most cases, but was only able to prove this theorem for $s \in (0, 1)$. The initial value problem (B) is in the same form as (A). The only difference is that the initial data is given on the line $x = x_0$. The coefficients to order k of $w(0, \zeta)$ are known from $u_k(x_0, \zeta + y_0)$. The same algorithms that are used to find the coefficients of $u_k(x, y)$ can be used to find the coefficients for the series of $w(\xi, \zeta)$.

Finally consider the problem

$$\frac{\partial v}{\partial x} = \underline{G}(v) \frac{\partial v}{\partial y}, \quad v(0, y) = \underline{f}(y), \quad (C)$$

where $\underline{G}(v) = \sum_{n=0}^{\infty} |a_n| v^n$ and $\underline{f}(y) = \sum_{n=0}^{\infty} |b_n| y^n$.

Theorem: If (x_0, y_0) is in the domain of convergence of the power series of $v(x, y)$, the solution to problem (C), expanded about the origin; and if (x_1, y_1) is in the domain of convergence of the power series of $u(x, y)$ expanded about the point (x_0, y_0) , then there exists an $s \in (0, 1)$ such that

$$\lim_{k \rightarrow \infty} w_{s,k}(\xi_0, \zeta_0) = u(x_1, y_1).$$

3. The Problem

The general equations of motion for an inviscid compressible flow in spherical coordinates are given by

$$\begin{aligned}
 & w \frac{\partial w}{\partial r} + \frac{u}{r} \frac{\partial w}{\partial \theta} + \frac{v}{r \sin \theta} \frac{\partial w}{\partial \phi} + \frac{1}{\rho} \frac{\partial p}{\partial r} - \frac{(u^2 + v^2)}{r} = 0, \\
 & w \frac{\partial u}{\partial r} + \frac{u}{r} \frac{\partial u}{\partial \theta} + \frac{v}{r \sin \theta} \frac{\partial u}{\partial \phi} + \frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \frac{(uw - v^2 \cot \theta)}{r} = 0, \\
 & (1) \quad w \frac{\partial v}{\partial r} + \frac{u}{r} \frac{\partial v}{\partial \theta} + \frac{v}{r \sin \theta} \frac{\partial v}{\partial \phi} + \frac{1}{\rho r \sin \theta} \frac{\partial p}{\partial \phi} + \frac{(vw + uv \cot \theta)}{r} = 0, \\
 & \frac{\partial}{\partial r} (\rho r^2 w \sin \theta) + \frac{\partial}{\partial \theta} (\rho r u \sin \theta) + \frac{\partial}{\partial \phi} (\rho r v) = 0,
 \end{aligned}$$

where $w = dr/dt$, $u = r d\theta/dt$, $v = r \sin \theta d\phi/dt$, p = pressure and ρ = density.

The first three equations are the equations for conservation of momentum; the fourth is the continuity equation. To these must be added the energy equation for a steady adiabatic flow; along a stream-line $u \partial S / \partial \theta + (v / \sin \theta) (\partial S / \partial \phi) = 0$, where S is the entropy. The equation of state for a polytropic gas is

$$(2) \quad p = A(S) \rho^\gamma,$$

where γ is the gas constant. S is removed from the energy equation using (2). This leaves five differential equations in five unknowns.

Inside the shock, the flow is considered conical, that is, the condition that all flow quantities are independent of the variable r . This assumption implies the following two equivalent statements. Along each ray through $r = 0$, the flow quantities are equal at every point. It is sufficient to solve the equations on the spherical surface $r = \text{constant}$, for the flow is similar on all spherical surfaces (actually identical except for a scaling factor).

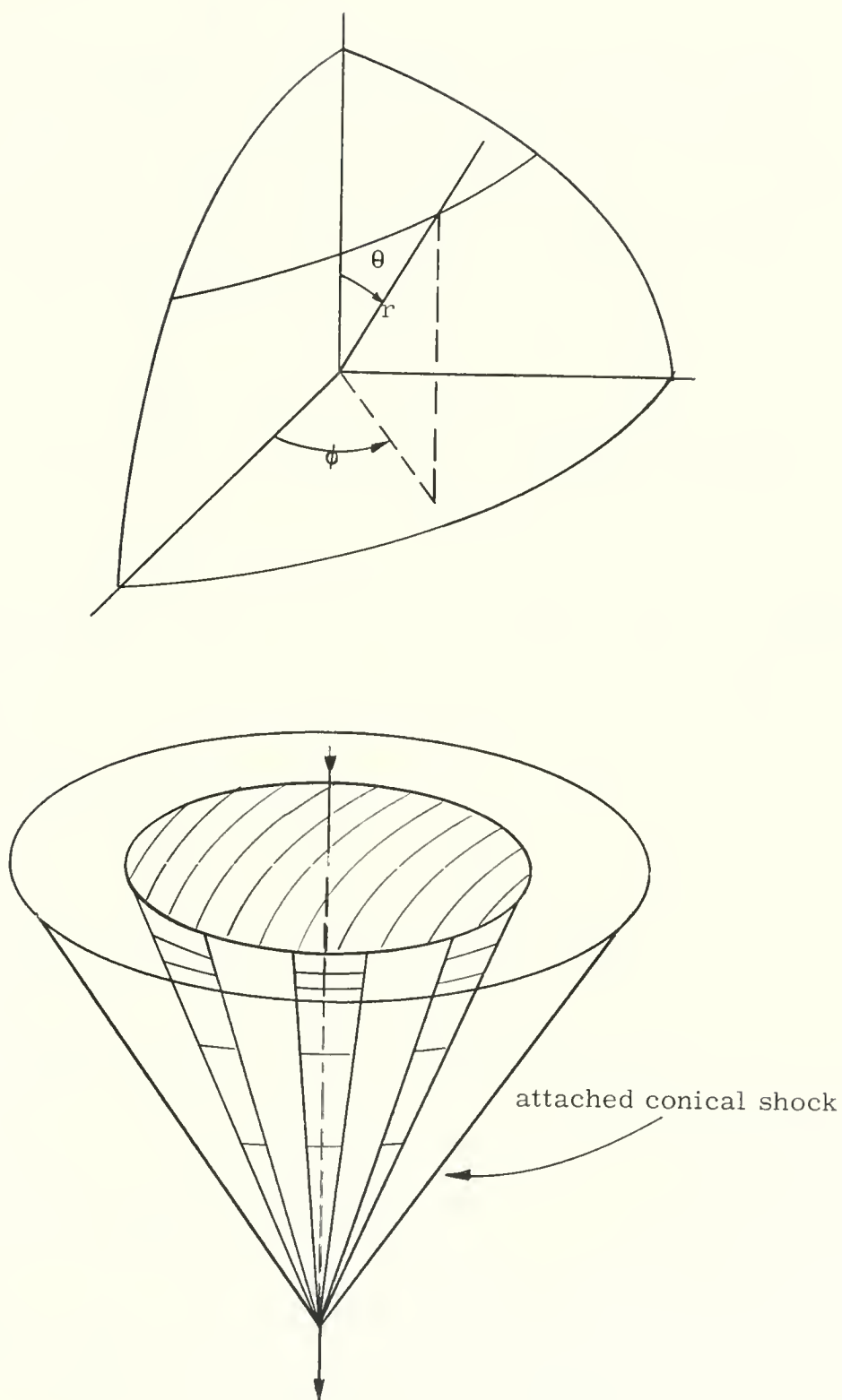


Figure 1

Let the shock be given by

$$(3) \quad g(\phi) = \theta \quad .$$

In order to use the Cauchy-Kowalewski method, the equations must be rewritten in terms of new coordinates such that the shock becomes the initial curve $\xi = 0$ (the shock is actually the cone $g(\phi) = \theta$, $r = r$, but we are only considering the intersection of this cone with the surface $r = \text{constant}$ so that the shock is represented by a curve).

Let

$$(4) \quad \xi = g(\phi) - \theta \quad , \quad \eta = \phi \quad .$$

The shock is now the line $\xi = 0$. The equations inside the shock become:

$$(5) \quad \begin{aligned} \frac{\partial p}{\partial \xi} &= \left\{ -[v\Omega - \frac{\gamma p}{\rho} g'] \frac{\partial p}{\partial \eta} + \gamma p \sin \theta \left\{ u \frac{\partial v}{\partial \eta} - v \frac{\partial u}{\partial \eta} + [u \cos \theta - w \sin \theta]u \right. \right. \\ &\quad \left. \left. + [g'w + v \cos \theta]v - 2\Omega w \right\} \right\} / \left\{ \Omega^2 - \frac{\gamma p}{\rho} (g'^2 + \sin^2 \theta) \right\} \quad , \\ \frac{\partial w}{\partial \xi} &= \frac{\left\{ -v \frac{\partial w}{\partial \eta} + (u^2 + v^2) \sin \theta \right\}}{\Omega} \quad , \\ \frac{\partial u}{\partial \xi} &= \frac{\left\{ -v \frac{\partial u}{\partial \eta} - uw \sin \theta + v^2 \cos \theta + \frac{\sin \theta}{\rho} \frac{\partial p}{\partial \xi} \right\}}{\Omega} \quad , \\ \frac{\partial v}{\partial \xi} &= \frac{\left\{ -v \frac{\partial v}{\partial \eta} - vw \sin \theta - uv \cos \theta - \frac{1}{\rho} \frac{\partial p}{\partial \eta} - \frac{g'}{\rho} \frac{\partial p}{\partial \xi} \right\}}{\Omega} \quad , \\ \frac{\partial \rho}{\partial \xi} &= \frac{\left\{ -v \frac{\partial \rho}{\partial \eta} + \frac{\rho}{\gamma p} [\Omega \frac{\partial p}{\partial \xi} + v \frac{\partial p}{\partial \eta}] \right\}}{\Omega} \quad , \end{aligned}$$

where $\Omega = vg' - u \sin \theta$, $g' = dg/d\eta$, $\sin \theta = \sin (g(\eta) - \xi)$, $\cos \theta = \cos (g(\eta) - \xi)$, $p = p(\xi, \eta)$, $\rho = \rho(\xi, \eta)$, $u = u(\xi, \eta)$, $v = v(\xi, \eta)$, $w = w(\xi, \eta)$. The characteristic equations for this system are

$$(\Omega d\eta - v d\xi)^3 \left\{ \Omega(v^2 - \frac{\gamma p}{\rho}) d\xi^2 + [\frac{\gamma p}{\rho}(2v \sin^2 \theta + v g'^2 + g'^2 \Omega - u g') - 2v \Omega^2] d\xi d\eta + \Omega(\Omega^2 - \frac{\gamma p}{\rho}(g'^2 + \sin^2 \theta)) d\eta^2 \right\} = 0, \quad (6)$$

where

$$\frac{d\xi}{dt} = \frac{1}{r \sin \theta} (v g' - u \sin \theta) \quad \text{and} \quad \frac{d\eta}{dt} = \frac{v}{r \sin \theta}. \quad (7)$$

The form (5) of the flow equations is valid so long as neither

(a) $\theta = 0$ nor

(b) $\Omega = 0$ nor

(c) $\Omega^2 - \frac{\gamma p}{\rho}(g'^2 + \sin^2 \theta) = 0$.

In case (a), $\theta = 0$ corresponds to the origin of the polar coordinates (θ, ϕ) on the spherical surface (this point will be referred to as the polar origin; θ and there ξ correspond to the radial component, ϕ and η refer to the angular one). The polar origin lies inside the body. If either case (b) or (c) should occur, the curve $\xi = \text{constant}$ would be characteristic, which can be seen by setting $d\xi = 0$ in (6). This would violate the conditions of the Cauchy-Kowalewski Theorem. However at the shock $\Omega \neq 0$ since $\Omega / \sqrt{\sin^2 \theta g + g'^2}$ is the normal component, N , of velocity (see below). $N = 0$ only in the case of a contact discontinuity; by the conservation of mass it cannot vanish otherwise. If (c) were the case, then

$$\frac{\left(\frac{\Omega}{\sqrt{\sin^2 \theta g + g'^2}} \right)^2}{\frac{\gamma p}{\rho}} = \frac{N^2}{\frac{\gamma p}{\rho}} = 1.$$

This is a contradiction since the normal component of velocity must be subsonic behind a stationary shock. The stagnation point on the body is an exceptional point for the solution, for there $N = 0$ and $d\xi = 0$,

so that it cannot be used as a point of continuation by the second method. In terms of u and v the flow was found to be subsonic between the shock and the body, so there are probably no exceptional points interior to the flow where a continuation would not be possible.

Figure 2a is a cross-section picture looking at the plane $\phi = \text{constant}$. Figure 2b is a drawing looking down on the spherical surface $r = \text{constant}$. From Figure 2c, $\sin \alpha$ and $\cos \alpha$ are readily found. Since $\theta = g(\phi)$, $d\theta = g' d\phi$,

$$(8) \quad \sin \alpha = \frac{g'}{\sqrt{\sin^2(g) + g'^2}}, \quad \cos \alpha = \frac{\sin(g)}{\sqrt{\sin^2(g) + g'^2}}$$

so that the components of velocity at the shock are

$$(9) \quad \begin{aligned} N &= -[(-u) \cos \alpha + v \sin \alpha] = \frac{u \sin(g) - v g'}{\sqrt{\sin^2(g) + g'^2}}, \\ T &= -(-u) \sin \alpha + v \cos \alpha = \frac{u g' + v \sin(g)}{\sqrt{\sin^2(g) + g'^2}}, \end{aligned}$$

where N is the normal component of velocity and T the tangential.

The conditions ahead of the shock are

$$(10) \quad \begin{aligned} p_O &= 1, \quad \rho_O = 1, \quad \gamma = 1.405, \quad \dot{M} = 5.4223, \quad Q = M \sqrt{\gamma}, \\ u_O &= -Q \sin \theta, \quad v_O = 0, \quad w_O = Q \cos \theta, \end{aligned}$$

$$B = \frac{2\gamma}{\gamma - 1} \frac{p_O}{\rho_O} + u_O^2 + v_O^2 + w_O^2, \quad M \text{ is the mach number.}$$

Based on these, we can find the initial values of the flow quantities behind the shock by the Rankine-Hugoniot jump conditions:

$$\begin{aligned}
\rho_1 &= \frac{\lambda u_o^2}{(\lambda + 1)(1 + (\frac{g'}{\sin g})^2) + u_o^2} , \\
p_1 &= \frac{\lambda \rho_1 - 1}{\lambda - \rho_1} , \\
u_1 &= u_o - \frac{\lambda - 1}{\lambda} \frac{u_o}{(1 + (\frac{g'}{\sin g})^2)} + \frac{\lambda + 1}{\lambda u_o} , \\
v_1 &= (u_o - u_1)(\frac{g'}{\sin g}) , \\
w_1 &= w_o ,
\end{aligned}
\tag{11}$$

where $\lambda = (\gamma + 1)/(\gamma - 1)$, $g' = dg/d\phi$. If θ and ϕ are considered as radial and angular components of polar coordinates, then g was chosen so that $\theta = g(\phi)$ would be the polar representation of an ellipse. The plane cross-section of the shock is given by

$$R = r \cos(a) \tan(g(\phi)) ,$$

where R is the radial distance measured on the plane surface; the plane is normal to the axis of the cone and passes through the point $(r=r, \theta=g(0)=a, \phi=0)$. In principle, any analytic noncharacteristic curve would do. Two problems were solved. In one case the ratio of b/a , the minor to the major axis, was 1 and in the other case $b/a = .7$. The curve g is given by

$$\theta = g(\phi) = \frac{b}{\sqrt{1 + (\frac{b^2}{a^2} - 1) \cos^2 \phi}} .
\tag{12}$$

The conditions ahead were normalized to make the Bernoulli constant equal to unity by

$$\begin{aligned}
p &= \frac{p_1}{B} , & \rho &= \rho_1 , & u &= \frac{u_1}{\sqrt{B}} , \\
v &= \frac{v_1}{\sqrt{B}} , & w &= \frac{w_1}{\sqrt{B}}
\end{aligned}
\tag{13}$$

(note $p = p(0, \eta)$, $\rho = \rho(0, \eta)$, etc.). The particular choice of M , γ and the normalization was taken in order to compare the circular case ($b/a = 1$) with Kopal [7]. The results of this calculation are shown in Table 1 in Section 8.

All the functions are considered as power series in two variables, ξ and η . The equations (5) establish the recursive relations between the coefficients, and the conditions (11) and (13), found from the Rankine-Hugoniot jump conditions, provide the starting values.

4. The Algorithms

Subroutines automatically carry out the multiplication, division, differentiation, and integration. These routines are based on the following algorithms, which express these operations as rational algebraic combinations of the coefficients.

Let P , Q , and R be power series in two variables, e. g.

$$P = \sum_{i=1} \sum_{j=1} P_{i,j} x^{i-1} y^{j-1} ,$$

etc. (This notation was chosen so that the formulas would be compatible with Fortran.)

If $R = P \cdot Q$, then the coefficient

$$R_{I,J} = \sum_{i=1}^I \sum_{j=1}^J P_{i,j} Q_{I-i+1, J-j+1} .$$

If $R = P/Q$, then

$$R_{I,J} = \frac{1}{Q_{1,1}} \left(P_{I,J} - \sum_{i=1}^{I-1} \sum_{j=1}^J R_{i,j} Q_{I-i+1, J-j+1} - \sum_{j=1}^{J-1} R_{I,j} Q_{1, J-j+1} \right) ,$$

where the first sum is dropped if $I = 1$, and the second if $J = 1$. If

$$Q = \frac{\partial P}{\partial x} \quad \text{or} \quad R = \frac{\partial P}{\partial y} ,$$

then

$$Q_{I,J} = I \cdot P_{I+1,J} , \quad R_{I,J} = J \cdot P_{I,J+1} .$$

If

$$Q = \int_0^x P \, dx \quad \text{or} \quad R = \int_0^y P \, dy ,$$

then

$$Q_{I,J} = \frac{P_{I-1,J}}{I-1} , \quad R_{I,J} = \frac{P_{I,J-1}}{J-1} .$$

The series for $\sin(g(\eta)-\xi)$ and $\cos(g(\eta)-\xi)$ were calculated by considering the functional relations for $\sin(A+B)$, $\cos(A+B)$. These allowed separation of these series into sums and products of series of a single variable. Other routines were written to calculate the sine, cosine, exponential, logarithm and power of a series of a single variable. These routines were based on recurrence relations by differentiating products of series. Leibnitz' rule gave the formula to express these derivations. Assume that P , Q , and R are series in one variable, e. g.

$$P = \sum_{i=1}^{\infty} P_i x^{i-1} ,$$

etc. (This notation was chosen so that the formulas would be compatible with Fortram.) Suppose

$$Q = P^\lambda$$

then

$$Q' P = \lambda Q P'$$

and

$$D^k(Q' P) = \lambda D^k(Q P') , \quad ' = \frac{d}{dx} , \quad D^k = \frac{d^k}{dx^k} ,$$

$$\sum_{i=0}^k \binom{k}{i} Q^{k-i+1} P^i = \lambda \sum_{i=0}^k \binom{k}{i} Q^{k-i} P^{i+1}$$

so that the coefficients

$$\begin{aligned} Q_1 &= P_1^\lambda, \\ Q_2 &= \frac{\lambda Q_1 P_2}{P_1}, \\ Q_3 &= \frac{\left\{ \frac{(\lambda-1)}{2} Q_2 P_2 + \lambda Q_1 P_3 \right\}}{P_1}, \\ &\vdots \\ Q_{k+1} &= \frac{\left\{ \frac{(\lambda-k+1)}{k} Q_k P_2 + \lambda Q_1 P_{k+1} + \sum_{i=2}^{k-1} \left[\frac{i(\lambda+1)-1}{k} \right] Q_{k+1-i} P_{i+1} \right\}}{P_1} \end{aligned}$$

($k = 3, 4, \dots$).

Suppose

$$Q = \exp \left\{ P \right\},$$

then

$$Q' = P' Q$$

and

$$D^k Q' = D^k (P' Q),$$

so that the coefficients

$$\begin{aligned} Q_1 &= e^{P_1}, \\ Q_2 &= P_2 Q_1, \\ &\vdots \\ Q_{k+1} &= \frac{P_2 Q_k}{k} + \frac{1}{k} \sum_{i=1}^{k-1} (i+1) P_{i+2} P_{k-i} \end{aligned} \quad (k = 2, \dots).$$

Suppose

$$Q = \log \left\{ P \right\} ,$$

then

$$PQ' = P'$$

so that the coefficients

$$\begin{aligned} Q_1 &= \log (P_1) , \\ Q_2 &= \frac{P_2}{P_1} , \\ &\vdots \\ Q_{k+1} &= \frac{\left\{ P_{k+1} - \frac{1}{k} \sum_{i=1}^{k-1} (k-i) Q_{k-i+1} P_{i+1} \right\}}{P_1} \quad (k = 2, 3, \dots). \end{aligned}$$

Suppose

$$Q = \sin (P) \quad \text{and} \quad R = \cos (P)$$

then

$$Q' = P' R \quad \text{and} \quad R' = -P' Q$$

so that the coefficients

$$\begin{aligned} Q_1 &= \sin (P_1) , & R_1 &= \cos (P_1) , \\ Q_2 &= R_1 P_2 , & R_2 &= -Q_1 P_2 , \\ &\vdots & & \\ Q_{k+1} &= R_1 P_{k+1} + \frac{1}{k} \sum_{i=1}^{k-1} (k-i) R_{i+1} P_{k-i+1} , \\ R_{k+1} &= -Q_1 P_{k+1} - \frac{1}{k} \sum_{i=1}^{k-1} (k-i) Q_{i+1} P_{k-i+1} \quad (k = 2, 3, \dots). \end{aligned}$$

If a general function of a function had entered into the equations (5) or (11), the problem would have been much more difficult, especially since

the coefficients for these functions must be recalculated each time another point of expansion is used. These routines were important because of the form of $g(\phi)$ in (12) and the appearance of $\sin(g(\phi)-\xi)$ in the equations (5) and (11).

5. The Body

After all the flow quantities have been computed, the body is approximated by two methods. A stream function does not exist for this flow, so the body is found from the velocity field which forms an envelope around the body. The normal component of velocity vanishes along the body. By locating the stagnation point along the major axis, the field line that passes through this point, perpendicular to the major axis, describes the body. Such a point exists by symmetry. The field line is made up of velocity vectors with components $d\xi$, $d\eta$ which are functions of u and v only (see equation (7)). In practice, the field cannot be found at the stagnation point, and a starting point, displaced a short distance (10^{-5} radians) along the perpendicular to the major axis, was used. This was justified since the stagnation point along the major axis is assumed to be a regular point, so that the velocity field has to be smooth there.

In the second method, the body was found by considering $A(S) = p/\rho^\gamma$. S , the entropy, is constant along a streamline. The body is a part of the dividing streamline. The body was found by calculating the entropy along a radial until the value was found to equal the entropy on the dividing streamline. The entropy increases monotonically from the shock to the body since the shock monotonically decreases in strength from the major axis to the minor axis. This calculation indicated that the body differed from the ellipse

$$\frac{x^2}{(.32882)^2} + \frac{y^2}{(.14226)^2} = 1$$

by less than .006 radians (see Figure 4).

6. The Consistency Checks

Several tests were used to determine the accuracy of the computation. These were also useful in determining where to choose the next point for continuation.

Bernoulli's equation was not used in the calculation of the flow, so it was available as a consistency check. Since the Bernoulli constant, B , is constant along a streamline, even through a shock, and since the flow outside the shock is constant, then B is constant everywhere in the flow. In the initial conditions B was normalized to equal one. So by computing $[2\gamma/(\gamma - 1)](p/\rho) + u^2 + v^2 + w^2$ at different points and comparing this with $B = 1$, a check was provided on the calculation. Except in the neighborhood of the body near the minor axis, the calculated value differed from $B = 1$ by at most 10^{-7} .

Another test consisted of comparing evaluation of the power series expanded about different points at common points where both series converged. As above, in the neighborhood of the body, near the minor axis, agreement was poor.

The routine that calculated the body from the velocity field was also used to follow other particle paths. The calculation of the entropy served as a check on this program and the results were good (5-place accuracy) except in the region mentioned above. In that region there are large gradients of entropy across the streamlines making the test unsuitable.

Significance arithmetic was used throughout the problem. The significance index was a good indicator for choosing the next point for continuation. When evaluating a series the index was examined. The index, rather than gradually decreasing as the point of evaluation moved

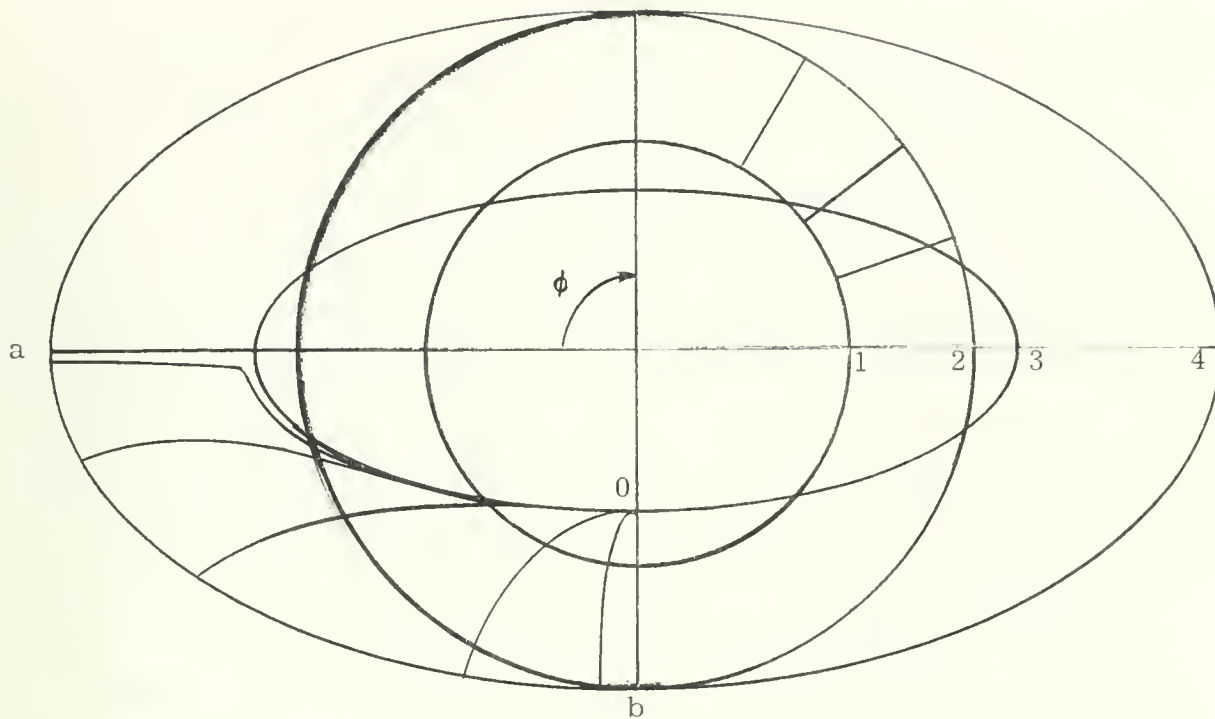


Figure 3

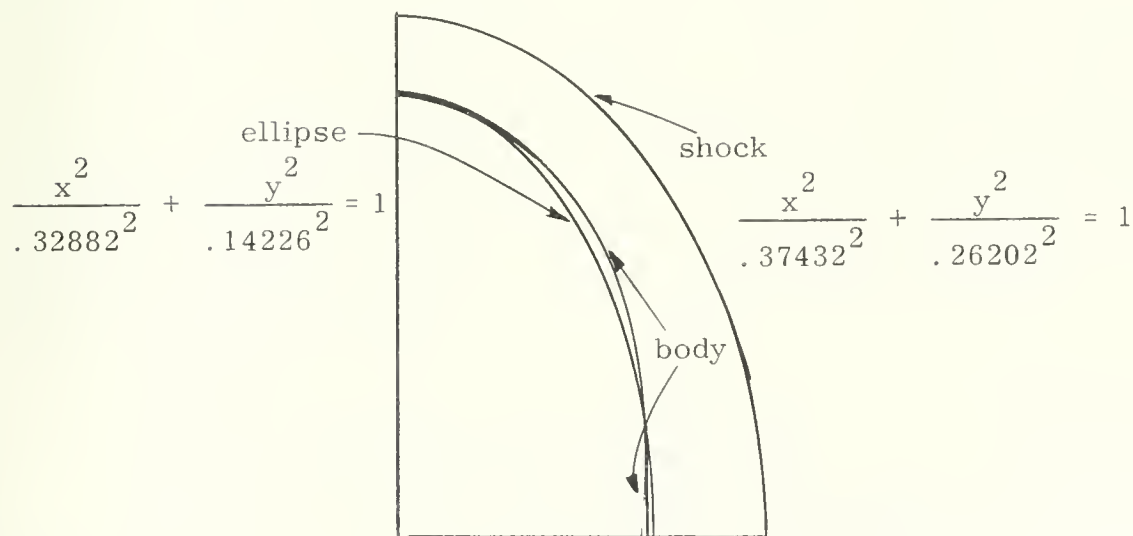


Figure 4

closer to the boundary of convergence, would fall rapidly near the boundary; it would stay constant inside the region of convergence.

Significance arithmetic on New York University's IBM 7090 is described thoroughly in [8] and [9]. The ordinary arithmetic of decimally represented numbers is carried out in what is called floating point arithmetic; this is without regard to accuracy or rounding. In the machine a number is represented by $2^p \cdot f$ where $0 \leq f < 1$ and f has 27 bits. Ordinarily when two numbers are added the last step consists of normalizing the answer by shifting the fraction f left until the leading bit is a one. The exponent is decreased by one for each place that f is shifted. In significance arithmetic this step is suppressed. Addition and subtraction are followed by rounding. In multiplication and division an effort is made to give the answer the same number of zeroes as the least significant operand. This is not always possible due to machine timing requirements, and so the fraction may carry an extra leading zero. A statistical decision is used in the part of the operation that adjusts the number of leading zeroes, accounting for the discrepancy. Rounding follows multiplication, but not after division, also due to timing. However a trap follows division which allows the programmer to use a subroutine to perform the rounding. This was not taken advantage of in this program. The significance index is 27 minus the number of leading zeroes in the fraction, f . A number with no significance would have an index of zero; a nonzero number with complete significance would have an index of 27.

7. Description of the Calculation

The problem was broken into several subproblems, each of which was solved by a separate calculation. The reason for this was that several of the calculations took up most of the storage space available in the memory; to combine them would have meant a loss in speed

and flexibility. Also many of the tests were qualitative, necessitating frequent interruption of the program. One of these tests determined the appropriate point for a continuation.

To start the problem, a point $(0, \eta_0)$ and the ratio of the minor to major axis is read into the machine and the series for the initial curve g is computed. This is an expansion in powers of $\eta - \eta_0$. At this time $\sin(g(\eta) - \xi)$ and $\cos(g(\eta) - \xi)$ are also calculated. These series were always calculated to 24-th order. The coefficients were stored on tape for the other programs. The expansions for sine and cosine were compared with the corresponding machine functions for several different values. These tests always showed agreement to at least seven places.

The next program calculated the expansions for p , ρ , u , v , w . The order of the series was read into the machine as well as whether or not the particular calculation was a continuation. The order was usually 22 at the shock, and one less for each continuation. The order occasionally had to be reduced for expansions near the body when the machine had an overflow from calculating a high order term. In the case of a continuation, the new point of expansion had to be given. Continuations were always along radial lines ($\eta = \text{constant}$) through the polar origin. It was also necessary to recalculate the series for the sine and cosine since the equations were written in terms of the new variable $\xi' = \xi - \xi_0$. This calculation was much simpler than the original calculation for the sine and cosine, since the functional relations for these could be used. The initial data were written into the program. At this juncture of the program, the operation for calculating the series for the flow variables is the same as the original expansions or the continuations. From the Rankine-Hugoniot conditions or by evaluating the series to be continued, the coefficients $p_{1,j}$, $\rho_{1,j}$, $u_{1,j}$, $v_{1,j}$, $w_{1,j}$ ($j = 1, \dots, k$) ($k - 1 = \text{the order}$) are available.

Assume that the coefficients, $p_{i,j}$, $\rho_{i,j}$, $u_{i,j}$, $v_{i,j}$, $w_{i,j}$ are known for all $i+j \leq k$, $i \leq I$. From these, the coefficients $(\partial p / \partial \eta)_{i,j}$, $(\partial \rho / \partial \eta)_{i,j}$, etc., of the series $\partial p / \partial \eta$, etc., are computed for $i+j \leq k-1$, $i \leq I$. The coefficients of every series on the right of (5) are known for $i+j \leq k-1$, $i \leq I$. Hence $(\partial p / \partial \xi)_{i,j}$ can be computed for $i+j \leq k-1$, $i \leq I$. This, then, gives $p_{i,j}$ for $i+j \leq k$, $i = I+1$ since $(\partial p / \partial \xi)_{i,j} = i p_{i+1,j}$. Similarly the coefficients for ρ , u , v , w are computed for $i+j \leq k$, $i = I+1$ and the process is repeated till all the coefficients are found.

After the calculation, these series are stored on tape. This program used up most of the memory, and could have been written to use much less storage. However, as it was coded, the program could have been converted to double precision and use the same amount of storage; this was not tried.

Another program evaluated the flow quantities, the Bernoulli constant, and the entropy along a radial line. This was used, not only to compute tables, but to find where continuations were necessary, and also for the test which compared power series in overlapping regions of convergence.

The routine which calculated the streamlines from the velocity field was based on the following second order scheme.

Suppose we are at the point (ξ_o, η_o) . Calculate the point (ξ_1, η_1) by

$$\xi_1 = \xi_o + \frac{d\xi_o}{\sqrt{d\xi_o^2 + d\eta_o^2}} \cdot \frac{\delta}{2} ,$$

$$\eta_1 = \eta_o + \frac{d\eta_o}{\sqrt{d\xi_o^2 + d\eta_o^2}} \cdot \frac{\delta}{2} .$$

The point (ξ, η) , the end point, is found by

$$\xi = \xi_0 + \frac{d\xi_1}{\sqrt{d\xi_1^2 + d\eta_1^2}} \cdot \delta, \quad \eta = \eta_0 + \frac{d\eta_1}{\sqrt{d\xi_1^2 + d\eta_1^2}} \cdot \delta,$$

$$d\xi_i = d\xi \left| \begin{array}{l} \xi = \xi_i \\ \eta = \eta_i \end{array} \right., \quad d\eta_i = d\eta \left| \begin{array}{l} \xi = \xi_i \\ \eta = \eta_i \end{array} \right. \quad (i = 0, 1);$$

$d\xi$ and $d\eta$ are defined in equation (7); δ is a constant. This calculation was done as a separate program. The coordinates of an initial point (ξ_1, η_1) are read into the machine, along with the precalculated series coefficients. At each successive point the flow variables and the entropy were calculated by summing the appropriate series. The program would calculate the streamline from (ξ_1, η_1) to (ξ_2, η_2) using expansions about a given η_0 , $\eta_1 < \eta_0 < \eta_2$. This program would determine the spacing of the expansion points $(0, \eta_0)$. The η_0 were chosen sufficiently close together so that the field would be continuous and smooth at (ξ_1, η_1) and (ξ_2, η_2) .

8. Results

Figure 3 illustrates the difference between the flows with $b/a = 1$, and $b/a = .7$. Curves 2 and 4 represent the shocks, 1 and 3 the respective bodies. The particle paths for the circular case are shown in the first quadrant. Each streamline is a dividing streamline. In contrast with this are the streamlines for the elliptical case shown in the third quadrant. The streamlines approach the body fast and all join at 0, which is a singular point. In the neighborhood of the body between $\eta = 1$ and $\eta = \pi/2$ the streamlines are bunched causing large gradients in the entropy; the gradients increased toward $\eta = \pi/2$. This result is illustrated in Figure 5. The entropy is plotted against the percentage of distance from the shock to the body. Each curve was found by

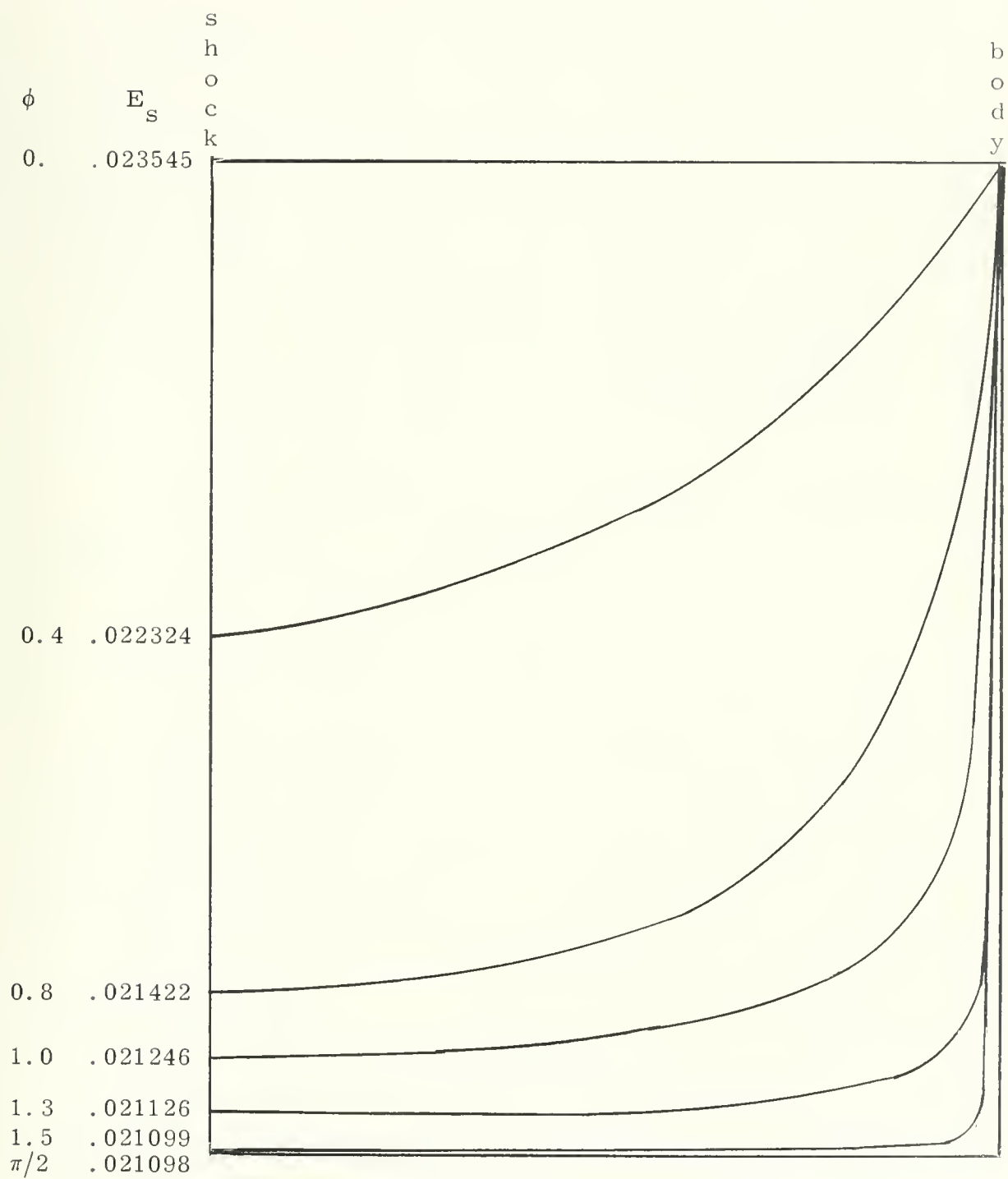


Figure 5

computing the entropy along the radial lines $\eta = \text{constant}$. The values η are indicated along with the values E_s of the entropy at the shock. At $\eta = 1.0$ the effects of the changes in entropy are apparent. Additional continuations could not improve the calculation near the body. Between $\eta = 1$ and the singularity at $\eta = \pi/2$ the body acted as if it were a line of singularities, and calculations were not accurate very near it. The expansion on the line $\pi/2$ nearest the singular point, if considered alone, would indicate that the singular point was a stagnation point and regular. However the velocity component v was of opposite sign from v calculated by neighboring expansions. Away from the neighborhood of the body, all the expansions were consistent, with high accuracy and agreement. As $\pi/2$ was approached along the body, both of these properties were lost.

Figure 4 shows the body as it was found by calculating the entropy. It is compared with the ellipse:

$$\frac{x^2}{.32882^2} + \frac{y^2}{.14226^2} = 1.$$

Table 1 illustrates the comparison with Kopal [7, p. 166]. The first column is the argument θ , given in degrees. The second and third columns show Kopal's calculation of the velocity components ($v \equiv 0$). The last four columns are calculations from this problem. B is the Bernoulli constant $[2\gamma/(\gamma - 1)](p/\rho) + u^2 + w^2$, $E = p/\rho^\gamma$.

Table 2 is a portion of the tests used for checking the power series agreement.

For the circular and the elliptical case, a line of singularities was discovered in the region outside the shock found by continuing the functions expanded on the shock. This was similar to the one discussed by Van Dyke in [1]. An attempt at solving the circular problem by using a bilinear mapping to move the line of singularities away was

unsuccessful. A different mapping might be more successful, but it also might be difficult to apply to the elliptical case; it could introduce complicated functions into the flow equations.

Solution by power series to the problem of conical flow involving an angle of attack should give some interesting results. The singularity, found on the body at $\eta = \pi/2$ in this problem, moves inside the flow for the problem with large yaw. There are also sonic lines in the flow.

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θ	u Kopal	w Kopal	u
15.013	-.14140	.89372	-.141398117
15.0	-.14102	.89376	-.141013078
14.5	-.12665	.89492	-.126642108
14.0	-.11280	.89597	-.112799935
13.5	-.09920	.89690	-.099197567
13.0	-.08567	.89770	-.085667565
12.5	-.07210	.89839	-.072090499
12.0	-.05837	.89896	-.0583673328
11.5	-.04440	.89941	-.0444039643
11.0	-.03011	.89973	-.0301010698
10.5	-.01535	.89993	-.0153441489
10.1	-.00313	.90000	-.00311757375
10.0	-.00000	.90000	.0000090209961

TABLE 1

θ	C	B	E
.37239	0	.99999994	.0234223269
	0	.99999994	.0234223641
.36239	0	.99999994	.0234347306
	0	.99999994	.0234347671
.35239	1	.99999994	.0234513789
	1	.999999985	.0234514169
.34239	1	.99999997	.0234744802
	1	.999999985	.0234745167
.33239	1	.99999994	.0235076219
	1	.99999994	.0235076576
.32439	1	1.000000016	.0235442042
	1	.99999994	.0235442936

TABLE 2A

w	B	E
.893723622	.99999994	.0210976347
.993755659	.99999994	.0210976191
.894922987	.99999994	.0210976191
.895967498	.99999994	.0210976250
.896892399	.999999985	.0210976243
.897699028	.99999994	.0210976250
.898387432	.99999994	.0210976243
.898956805	.999999985	.0210976236
.899405435	.999999970	.0210976236
.899730816	.999999985	.0210976236
.899929494	.99999994	.0210976236
.899994195	.999999985	.0210976191
.899996921	.999999911	.0210976109

(Table 1)

u	v	w
-.131663367	.0214699909	.861886069
-.131663367	.0214702591	.861885622
-.108251534	.0248421535	.863083020
-.108251497	.0248424381	.863082573
-.085240304	.0295664996	.864042416
-.085240133	.0295668423	.864041969
-.0619579077	.0363407657	.864758521
-.0619578660	.0363413617	.864758074
-.0379564941	.0463652641	.865212321
-.0379568398	.0463660762	.865211874
-.0180792689	.0582596824	.865355954
-.0180794001	.0582522273	.865355656

(Table 2A)

θ	C	B	E
.33626	0	.999999985	.0220010728
	0	.999999994	.0220010839
.31626	0	.999999985	.0220816396
	0	.999999985	.0220816530
.29626	1	.999999925	.0222089961
	1	.999999994	.0222090080
.27626	1	.999999994	.0224213980
	1	.999999994	.0224213697
.25626	3	.999999994	.0228058599
	1	.999999994	.0228055552
.24026	3	.99999726	.0233712912
	2	.999999806	.0233714990

TABLE 2B

θ	C	B	E
.26529	0	.999999994	.0211160667
	0	.999999994	.0211160667
.24129	1	1.00000001	.0211177625
	1	.999999985	.0211177647
.21729	1	.999999994	.0211208947
	1	.999999970	.0211208962
.19329	2	.999999994	.0211274773
	2	1.00000001	.0211274780
.16929	2	.999999994	.0211465694
	2	.999999994	.0211465588
.15129	3	1.00000001	.0212243430
	3	1.00000009	.0212236516

TABLE 2C

u	v	w
-.159879729	.0523717880	.873486832
-.159879953	.052372162	.873486608
-.142547742	.0600173146	.876485497
-.142547995	.0600176543	.876485273
-.127483673	.067727171	.879086256
-.127479143	.067721389	.879086003
-.113500647	.074554712	.881223708
-.113512471	.074529886	.881223485
-.099161476	.079544358	.882644162
-.099299692	.079506494	.882645726
-.086753845	.082079887	.882743970
-.087303072	.082002178	.882752746

(Table 2B)

u	v	w
-.143219933	.0112342335	.892936304
-.143219933	.0112342350	.892936304
-.115017064	.0134353504	.896030501
-.115017317	.0134352826	.896030486
-.088149212	.0152707033	.898464277
-.088148281	.0152696297	.898464262
-.0617579401	.0168264821	.900253162
-.0617364049	.0168320313	.900253013
-.0354221612	.0178162627	.901378810
-.0350821376	.0179187737	.901375771
-.0169630229	.0190556735	.901629582
-.0136737871	.0167032823	.901602358

(Table 2C)

θ	C	B	E
.26211	0	.99999994	.0210980795
	0	1,00000001	.0210980795
.24011	1	.99999994	.0210981175
	1	.99999994	.0210981183
.21811	1	.999999970	.0210981823
	1	.999999985	.0210981838
.19611	2	.99999994	.0210983060
	2	.99999994	.0210983090
.17411	2	.99999994	.0210986048
	2	.99999994	.0210986122
.15211	3	.999999985	.0210999608
	3	.999999955	.0210999362

TABLE 2D

Explanation of Tables 2

These tables show a portion of the tests on the agreement of the power series at common points in regions where both series converge. The evaluations are along the radial lines $\eta = \text{constant}$. The first column gives the coordinate along the radial line in radians, the next indicates which continuation of the series. 0 indicates a series which is expanded about a point on the shock. Column B is the evaluation of $\frac{-2\gamma p}{(\gamma-1)\rho} + u^2 + v^2 + w^2$, column E is p/ρ^γ and the last three columns are the velocity components.

Table 2A compares expansions about $\eta = 0$, $\eta = .2$ along the radial line $\eta = .1$. The top entries are from $\eta = 0$.

Table 2B compares expansions about $\eta = .4$, $\eta = .6$ along

u	v	w
-.141445637	.00179509670	.893703550
-.141445629	.00179509513	.893703550
-.114868477	.00213072166	.896519005
-.114868671	.00213076428	.896519020
-.089704834	.00241031542	.898767680
-.089706264	.00241206735	.898767710
-.065109186	.00264719680	.900469899
-.065123819	.00266127951	.900470048
-.0405864626	.00279437006	.901632369
-.0407251745	.00289380185	.901633725
-.0146235406	.00316989109	.902243361
-.0160383821	.00261764005	.902256638

(Table 2D)

the radial line $\eta = .5$. The top entries are from $\eta = .4$.

Table 2C compares expansions about $\eta = 1.3$, $\eta = 1.4$ along the radial line $\eta = 1.35$. The top entries are from $\eta = 1.3$.

Table 2D compares expansions about $\eta = 1.5$, $\eta = \pi/2$ along the radial line $\eta = 1.5353981629$. The top entries are from $\eta = 1.5$.

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